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THE FAR FIELD OF A MOVING OSCILLATING SOURCE IN THE CASE OF RESONANCE†

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The field excited by a moving oscillating source in a two-dimensional linear medium with dispersion (for example, a source of surface waves) is considered. It is assumed that the velocity of the source is equal to the group velocity corresponding to its oscillation frequency (taking the Doppler shift into account), i.e. resonance occurs. The asymptotic form of the wave field in the far zone for long time t is described. In particular, in the neighbourhood of the zero there is a resonance zone in which the wave field is of the order of unity or higher and the size of which increases as $t^{2/3}$ in critical directions, i.e. in directions perpendicular to the dispersion curve at its point of self-intersection. In directions which differ from the critical direction, the size of the resonance zone increases as $t^{1/2}$. The case of a degenerate stationary point of the dispersion function is also considered. A sharper resonance then occurs and the field increases as $t^{1/6}$. The three-dimensional problem is briefly considered. © 1998 Elsevier Science Ltd. All rights reserved.

In the general theory of fields, excited by oscillating wave sources, it is assumed [1, 2] that steady oscillations excited by a source which oscillates at a frequency ω_0, are described by the Fourier integral

$$q = \exp(i\omega_0 t)\iint \frac{F(\lambda, \mu)\exp(-i(\lambda x + \mu y))}{B(\omega_0, \lambda, \mu)} d\lambda d\mu \tag{0.1}$$

(the two-dimensional problem is considered). Here $B(\omega, \lambda, \mu) = 0$ is the dispersion equation. In other words, it is assumed that the wave field in the medium considered can have the form of a plane wave $\exp i(\omega t - \lambda x - \mu y)$ only if $B(\omega, \lambda, \mu) = 0$.

If the oscillating source moves uniformly and rectilinearly with a velocity $\mathbf{V} = (V_x, V_y)$, the steady field in a system of coordinates $\hat{x} = x - V_x t; \hat{y} = y - V_y t$, moving together with the source, is also described by integral (0.1), if x and y are replaced by $\hat{x}, \hat{y}$ in the exponent of the exponential function, and the dispersion function $B(\omega_0, \lambda, \mu)$ is replaced by $\hat{B}(\omega_0, \lambda, \mu) = B(\omega_0 - V_x\lambda - V_y\mu, \lambda, \mu)$.

When evaluating integral (0.1) the problem of regularization arises, i.e. the problem of how to understand integral (0.1) in the neighbourhood of the zeros of the function $B(\omega_0, \lambda, \mu)$. Natural physical considerations, which reduce to the fact that the inflow of energy from infinity is excluded, lead to the radiation principle, according to which integral (0.1) must be understood as the limit [1, 2]

$$q = \lim_{\varepsilon \to +0} \exp(i\omega_0 t)\iint \frac{F(\lambda, \mu)\exp - i(\lambda x + \mu y))}{B(\omega_0 - i\varepsilon, \lambda, \mu)} d\lambda d\mu \tag{0.2}$$

Below we consider the case when the curve $B = 0$ has singular points at which the partial derivatives of the function B with respect to λ and μ vanish. The integral on the right-hand side of (0.2) then increases as $\ln \varepsilon$ as $\varepsilon \to 0$. We will show that in this case steady oscillations do not exist in general. If the oscillating source is disconnected and motion begins at a certain instant of time t_0, then, as $t \to \infty$, the field does not tend to an expression of the form $\exp(i\omega_0 t)q(\hat{x}, \hat{y})$, but increases logarithmically. The reason is that when there is a singular point $\lambda = \lambda_0$, $\mu = \mu_0$ on the dispersion curve the group velocity of the oscillations with frequency ω_0 is zero (in a moving system of coordinates $\hat{x}, \hat{y}$ and taking the Doppler frequency shift into account). Hence, at a fixed point of observation $\hat{x} = \text{const}, \hat{y} = \text{const}$ at a certain $t = t_1 > t_0$, the oscillations excited for all t in the interval $t_0 < t < t_1$ arrive with the same phase and are summed in modulus, i.e. the phenomenon of resonance occurs. Here, as will be shown below, for any medium and any point $\lambda = \lambda_0$, $\mu = \mu_0$ we can choose the velocity $\mathbf{V}$ of motion of the source and the frequency ω_0 of its oscillations such that this point is a singular point of the curve $\hat{B} = 0$ and the phenomenon of resonance occurs.

The motion of an oscillating source of surface waves with a resonance velocity has been considered in the case when the spectral density $F(\lambda, \mu)$ at the singular point $\lambda = \lambda_0$, $\mu = \mu_0$ of the dispersion curve vanishes [3]. No storage effects therefore arise here and the field approaches a finite limit as $t \to \infty$.

Below we find the asymptotic form of the far field for large values of t in the case of resonance and we write out expressions for the logarithmically increasing field component. The three-dimensional problem is briefly

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considered. In this case the field approaches a finite limit as $t \to \infty$, but, nevertheless, the effects of resonance and field storage lead to a qualitative change in the behaviour of the field in the far zone.

1. INTEGRAL REPRESENTATION OF THE FIELD OF A MOVING SOURCE

We will consider a medium in which the field of the source q, with a time-dependence of the form $\delta(t)$ has the form

$$q(t, x, y) = \iint F(\lambda, \mu)\exp i(-\lambda x - \mu y + t\omega(\lambda, \mu))d\lambda d\mu \tag{1.1}$$

The functions $F(\lambda, \mu)$ and $\omega(\lambda, \mu)$ depend on the problem considered. For example, in the Cauchy–Poisson problem for surface waves on deep water ($q(t, x, y)$ is the height of the free surface) Green's function consists of two terms of the form (1.1) with $\omega = \pm\sqrt{(kg + k^3\gamma)}$, where $k = \sqrt{(\lambda^2 + \mu^2)}$, g is the acceleration due to gravity and γ is the surface tension coefficient.

Green's function for the field of internal gravitational waves can be represented in the form of the sum of modes; the field of the nth mode can also be represented in the form of the sum of two terms of the form (1.1) with $\omega = \pm\omega_n(k)$ and $F = \text{const}\ \varphi_n(k, z)\varphi_n(k, z_0)\omega_n(k)k^{-2}$, where $\omega_n(k)$ and $\varphi_n(k, z)$ are the eigenvalues and eigenfunctions of the vertical eigenvalue problem [4]

$$\varphi'' + k^2\omega^{-2}(N^2(z) - \omega^2)\varphi = 0$$

with zero boundary conditions on the surface $z = 0$ and on the bottom $z = -H$ and with eigenfunctions normalized with weight $N^2(z)$. Here k is the free parameter and ω is the spectral parameter, the square of the Väisälä–Brunt frequency $N^2(z) = -g\rho_0^{-1}(z)d\rho_0/dz$.

Hence, in the three-dimensional problem of the propagation of internal gravitational waves in a layer $-H \leqslant z \leqslant 0$ of stratified liquid, the depth z occurs as a parameter. The results obtained below on the growth of the field hold for any characteristic of the field of internal gravitational waves in the integral representation (1.1) in which the spectral density $F(\lambda, \mu)$ does not vanish at the singular point λ_0, μ_0 of the dispersion curve $B(\lambda, \mu, \omega_0)$ and, in particular, for the elevation.

The field $W(t,x,y)$ of the source, moving with velocity $\mathbf{V} = (V_x, V_y)$, beginning at $t = 0$, and oscillating with a frequency ω_0, can be written in the form

$$W(t, x, y) = \exp(i\omega_0 t)\hat{W}(t, \hat{x}, \hat{y});$$

$$\hat{W}(t, \hat{x}, \hat{y}) = \int_0^t d\tau \iint_{-\infty}^{\infty} F(\lambda, \mu)\exp(i(-\lambda\hat{x} - \mu\hat{y} + \hat{\omega}(\lambda, \mu)\tau))d\lambda d\mu$$

$$\hat{\omega}(\lambda, \mu) = \omega(\lambda, \mu) - \lambda V_x - \mu V_y - \omega_0$$

where $\hat{x} = x - V_x t$, $\hat{y} = y - V_y t$ is a system of coordinates moving together with the source.

We will further consider the asymptotic form of W for large t, $r = \sqrt{(\hat{x}^2 + \hat{y}^2)}$. The singular points of the functions $\hat{x}(\lambda, \mu)$ and $F(\lambda, \mu)$ make a contribution to this asymptotic form. For example, for surface waves the function $\omega(\lambda, \mu)$ behaves as $\sqrt{k}$ when $k = \sqrt{(\lambda^2 + \mu^2)} \to 0$; for internal gravitational waves this function behaves as k as $k \to 0$, while the amplitude factor $F(\lambda, \mu)$ behaves as k^{-1}.

It can be shown that the contribution of the singular point $\lambda = \mu = 0$ to the asymptotic form of the far field for fixed t and $r \to \infty$ is of the order of $r^{-3/2}$ for surface waves and r^{-1} for internal waves. We will dwell on the contribution to the asymptotic form of $\hat{W}$ of such singular points and consider the terms in the asymptotic form governed by the singular points of the surface $\hat{\omega} = 0$, i.e. the values of λ, μ for which $\hat{\omega}$ is a regular function but $\hat{\omega} = \partial\hat{\omega}/\partial\lambda = \partial\hat{\omega}/\partial\mu = 0$.

As was stated above, for any λ_0, μ_0 we can choose a velocity $\mathbf{V}$ of the source and the frequency of its oscillations ω_0 such that the point λ_0, μ_0 is a singular point of the surface $\hat{\omega} = 0$. To do this we must put

$$V_x = \frac{\partial\omega}{\partial\lambda}, \quad V_y = \frac{\partial\omega}{\partial\mu}; \quad \omega_0 = \omega(\lambda_0, \mu_0) - \lambda_0 V_x - \mu_0 V_y$$

where the derivatives $\partial\omega/\partial\lambda$ and $\partial\omega/\partial\mu$ are taken at the point λ_0, μ_0.

We put $\lambda = \lambda_0 + \eta$, $\mu = \mu_0 + \zeta$. The expansion in powers of η, ζ has the form

$$\hat{\omega}(\lambda_0+\eta,\ \mu_0+\zeta)=\omega_2(\eta,\ \zeta)+\omega_3(\eta,\ \zeta)+O(\eta^2+\zeta^2)^2 \tag{1.2}$$

$$\omega_2(\eta,\ \zeta)=A\eta^2+B\zeta^2;\quad \omega_3(\eta,\ \zeta)=C\eta^3+D\eta^2\zeta+E\eta\zeta^2+F\zeta^3$$

(if necessary we rotate the λ, μ axes). Changing to the integration variables $\eta = \lambda - \lambda_0$, $\zeta = \mu - \mu_0$ we obtain

$$\hat{W}(t,\ \hat{x},\ \hat{y})=\exp i(-\lambda_0\hat{x}-\mu_0\hat{y})\tilde{W}(t,\ \hat{x},\ \hat{y}) \tag{1.3}$$

$$\tilde{W}(t,\ \hat{x},\ \hat{y})=\int_0^t Jd\tau$$

$$J=\iint_{-\infty}^{\infty} G(\eta,\ \zeta)\exp(i(-\eta\hat{x}-\zeta\hat{y}+\tilde{\omega}(\eta,\ \zeta)\tau))d\eta d\zeta \tag{1.4}$$

$$\tilde{\omega}(\eta,\ \zeta)=\hat{\omega}(\lambda_0+\eta,\ \mu_0+\zeta),\quad G(\eta,\ \zeta)=F(\lambda_0+\eta,\ \mu_0+\zeta)$$

We will assume that the stationary point $O = (0, 0)$ of the function $\tilde{\omega}(\eta, \zeta)$ is non-degenerate, i.e. $A \neq 0$, $B \neq 0$. Using the ordinary technique of the expansion of unity, we will assume that $G(\eta, \zeta)$ is an infinitely differentiable finite function and that, in the region supp G, the function $\tilde{\omega}(\eta, \zeta)$ is analytic and has a unique stationary point O.

If $AB > 0$, then O is an isolated point of the set $\Omega : \tilde{\omega}$. If $AB < 0$, then O is a point of self-intersection of the curve Ω; $\tilde{\omega}(\eta, \zeta) = 0$ we will assume in this case that the branches of Ω have non-zero curvature at 0, i.e. that the cubic polynomial $\omega_3(\eta, \zeta)$ does not vanish on the tangents $\zeta = \pm\eta\sqrt{(-B/A)}$ to the branches of Ω.

2. THE ASYMPTOTIC FORM OF INTEGRAL (1.4) WHEN $r = \sqrt{(\hat{x}^2 + \hat{y}^2)} \gg 1$, $t = O(r)$

This asymptotic form is determined, first, by the stationary points of the phase function in triple integral (1.4) and, second, by the asymptotic form of the integral J as $\tau \to \infty$. The stationary points of the phase function $\Phi(\tau, \eta, \zeta) = -\eta\hat{x} - \zeta\hat{y} + \tilde{\omega}(\eta, \zeta)\tau$ are specified by the equations $\Phi'_\eta = \Phi'_\zeta = \Phi'_\tau = 0$, whence $\tilde{\omega}'_\eta = \hat{x}/\tau$, $\tilde{\omega}'_\zeta = \hat{y}/\tau$, $\tilde{\omega} = 0$. It follows from these equations that for specified $\hat{x}$, $\hat{y}$ the stationary point $P = (\eta, \zeta)$ is a point on the curve Ω at which grad $\tilde{\omega}$ is parallel to the vector $(\hat{x}, \hat{y})$. If these vectors are directed opposite to one another, the value of τ at the stationary point of the phase function is negative and there are no stationary points in the region of integration with respect to τ in (1.4). If the vectors $(\hat{x}, \hat{y})$ and grad $\tilde{\omega}$ are in the same direction, the value of τ at the stationary point is equal to (r, φ are polar coordinates in the $\hat{x}, \hat{y}$ plane)

$$\tau_0=r/\lvert\operatorname{grad}\tilde{\omega}(P)\rvert;\quad (\hat{x}=r\cos\varphi,\ \hat{y}=r\sin\varphi) \tag{2.1}$$

If the origin of coordinates O is a point of self-intersection of the dispersion curve Ω (i.e. in expansion (1.2) $AB < 0$) and the direction φ approaches the direction of the normal to some branch of the curve Ω at the point O (i.e. $\operatorname{tg}\varphi \to \pm\sqrt{(-B/A)}$), then $P \to O$, $\lvert \operatorname{grad} \tilde{\omega}(P) \rvert \to 0$ and $\tau_0/r \to \infty$. We will consider this case below, but we will confine ourselves now to directions φ which differ from $\pm\operatorname{arctg}\sqrt{(-B/A)}$, $\pi \pm \operatorname{arctg}\sqrt{(-B/A)}$ and to values $t = O(r)$.

Under these conditions the asymptotic form of integral (1.4) is determined, first, by the contribution U_1 of the stationary point (τ_0, P) (if this point falls in the region of integration), i.e. when $t > \tau_0 > 0$ and, second, by the contribution U_2 of the boundaries of the integration region, i.e. the planes $\tau = 0$ and $\tau = t$. However, the plane $\tau = 0$ makes no contribution to the asymptotic form of the integral (1.4), since for large values of r and fairly small values of τ in the region supp $G(\eta, \zeta)$ the phase function in the inner integral in (1.4) has no stationary points and the integral decreases more rapidly for any power of r.

U_1 and U_2 are calculated by the stationary-phase method. The contribution of the stationary point (τ_0, P) of the phase function is

$$U_1=\sqrt{\frac{R}{r}}\frac{(2\pi)^{3/2}G(P)}{\lvert\operatorname{grad}\tilde{\omega}\rvert}\exp i\left(-\eta_P\hat{x}-\zeta_P\hat{y}-\frac{\pi\delta}{4}\right) \tag{2.2}$$

Here R is the radius of curvature of the curve Ω at the point P, $\delta = 1$, if this curve is convex in the neighbourhood of P (i.e. if $\tilde{\omega}''_{\sigma\sigma} > 0$, where $d/d\sigma$ is the differentiation in a direction tangential to Ω at the point P) and $\delta = -1$ otherwise.

The contribution of the boundary of the integration region $\tau = t$ to integral (1.4) is

$$U_2 = \frac{-2\pi i G(Q) \exp i(-\eta_Q \hat{x} - \zeta_Q \hat{y} + t\tilde{\omega}(Q) + \pi\gamma/2)}{t\tilde{\omega}(Q)\sqrt{|D|}}$$

$$D = \tilde{\omega}''_{\eta\eta}(Q)\tilde{\omega}''_{\zeta\zeta}(Q) - (\hat{\omega}''_{\eta\zeta}(Q))^2$$

$\gamma = \operatorname{sign} \tilde{\omega}''_{\eta\eta} = \operatorname{sign} \tilde{\omega}''_{\zeta\zeta}$ when $D > 0$ and $\gamma = 0$ when $D < 0$.

The point $Q = (\eta_Q, \zeta_Q)$ is found from the equations

$$\tilde{\omega}'_\eta = \hat{x}/t, \quad \tilde{\omega}'_\zeta = \hat{y}/t \tag{2.3}$$

Note that if the region supp G is sufficiently small, the solution of Eqs (2.3) in it is unique.

The asymptotic form of integral (1.4) when $r \gg 1$, $t = O(r)$ has the form

$$U = U_1\chi(t|\operatorname{grad}\tilde{\omega}| - r) + U_2 \tag{2.4}$$

where the function $\chi(t|\operatorname{grad}\tilde{\omega}| - r) = \chi(t/\tau_0 - 1) = 1$ when $0 < \tau_0 < t$ (i.e. when the point (τ_0, P) lies in the integration region) and $\chi = 0$ otherwise.

Hence, the field U considered is a wave propagating in the direction φ with velocity $\upsilon = |\operatorname{grad}\tilde{\omega}(P)|$; in front of the wave front $r = \upsilon t$ the field U is identical with U_2 and is of the order of t^{-1}, while behind the wave front the principle term of the asymptotic form of U is identical with U_1, is of the order of $r^{-1/2}$ and is independent of t. If $r \to \upsilon t$, the point Q approaches P, $\tilde{\omega}(Q) \to 0$, $U_2 \to \infty$ and the asymptotic form (2.4) becomes unsuitable. In this case the stationary point (τ_0, Q) of the phase function turns out to be close to the boundary $\tau = t$ of the integration region. Hence, the asymptotic form of integral (1.3), used in the neighbourhood of the wave front, can be expressed in terms of a Fresnel integral [5].

3. REDUCTION OF INTEGRAL (1.4) TO A SINGLE INTEGRAL WHEN t, $r \gg 1$, $r/t \to 0$

As t gradually increases, when $r/t \to 0$, the solution Q of system (2.3) approaches the stationary point of the function $\tilde{\omega}(\eta, \zeta)$. If at this point the function $\tilde{\omega}$ were not zero (i.e. if the curve Ω did not have a point of self-intersection in the region supp G), then as $t \to \infty$ the function U_2 would approach zero. In this case integral (1.4) approaches a finite limit as $t \to \infty$; its asymptotic form in the far zone is identical with U_1 and is determined by the point P on the curve Ω at which grad $\tilde{\omega}$ is parallel to the vector $\mathbf{r} = (\hat{x}, \hat{y})$ and is directed towards the same side as this vector. This asymptotic form is identical with the expression obtained previously in [1, 2].

In the problem considered $\tilde{\omega} = 0$ at the stationary point and the function $\tilde{\omega}(Q)$ approaches zero when $r/t \to 0$ as r^2/t^2. Hence, $U_2 \to \infty$ as $t \to \infty$ and, obviously, the asymptotic form (2.4) becomes inapplicable.

In order to obtain the asymptotic form $W(t, \hat{x}, \hat{y})$ as $t \to \infty$, we will obtain the asymptotic form of the integral J in (1.4) for large τ, i.e. when $r \gg 1$, $r/\tau \ll 1$, for which we will use the stationary-phase method. The stationary point Q in this integral has the coordinates

$$\begin{aligned} \eta = \eta_Q &= \frac{\xi}{2A} - \frac{3C\xi^2}{8A^3} - \frac{D\xi\nu}{4A^2B} - \frac{E\nu^2}{8AB^2} + O\left(\frac{r^3}{t^3}\right) \\ \zeta = \zeta_Q &= \frac{\nu}{2B} - \frac{D\xi^2}{8A^2B} - \frac{E\xi\nu}{4AB^2} - \frac{3F\nu^2}{8B^3} + O(r^3/t^3) \end{aligned} \tag{3.1}$$

where A, B, C, D, E and F are the coefficients of expansion (1.2), $\xi = \hat{x}/\tau$ and $\nu = \hat{y}/\tau$.

Evaluation of the inner integral J in (1.3) for large τ gives

$$J \approx C(Q) \exp i\Phi(r, \varphi, \tau)$$

$$C(Q) = \frac{2\pi G(Q)}{\tau\sqrt{|D(Q)|}} \exp i\left(\frac{\pi}{4}(\operatorname{sign} A + \operatorname{sign} B)\right), \quad D(Q) = \tilde{\omega}''_{\eta\eta}(Q)\tilde{\omega}''_{\zeta\zeta}(Q) - (\tilde{\omega}''_{\eta\zeta}(Q))^2$$

$$\Phi(r, \varphi, \tau) = -\eta_Q r\cos\varphi - \zeta_Q r\sin\varphi + \tilde{\omega}(Q)\tau$$

It follows from (1.2) and (1.3) that as $\tau \to \infty$

$$\Phi(r, \varphi, \tau) = r^2\Phi_2(\varphi)\tau^{-1} + r^3\Phi_3(\varphi)\tau^{-2} + O(r^4/\tau^3)$$

$$\Phi_2(\varphi) = -\omega_2\left(\frac{\cos\varphi}{2A}, \frac{\sin\varphi}{2B}\right), \quad \Phi_3(\varphi) = \omega_3\left(\frac{\cos\varphi}{2A}, \frac{\sin\varphi}{2B}\right)$$

where r, φ are polar coordinates.

We expand the functions $G(Q)$ and $\sqrt{(|D(Q)|)}$ in series in inverse powers of τ as $\tau \to \infty$

$$G\left(\frac{r}{\tau}\right) = G(Q) = G(0, 0) + G_1\frac{r}{\tau} + \ldots, \quad \sqrt{\left|D\left(\frac{r}{\tau}\right)\right|} = \sqrt{|D(Q)|} = 2/\sqrt{|AB|} + D_1\frac{r}{\tau} + \ldots \tag{3.2}$$

In order to confine ourselves in integral (1.4) to values of τ for which the asymptotic form (3.1)–(3.2) is applicable, we will use the expansion of unity. We will choose two fairly large constants C_1, C_2 ($C_1 < C_2$) and infinitely differentiable functions $h(r/\tau)$ and $g(r/\tau)$, for which

$$h(r/\tau) + g(r/\tau) = 1; \quad h(r/\tau) = 0 \text{ when } \tau < C_1 r, \quad g(r/\tau) = 0 \text{ when } \tau > C_2 r \tag{3.3}$$

Then, when $t > C_2 r$, integral (1.3) can be written in the form

$$\tilde{W}(t, r, \varphi) = W_1 + W_2$$

$$W_1 = W_1(r, \varphi) = \int_0^\infty g\left(\frac{r}{\tau}\right)J\left(\frac{r}{\tau}, \varphi\right)d\tau, \quad W_2 = W_2(t, r, \varphi) = \int_0^t h\left(\frac{r}{\tau}\right)J\left(\frac{r}{\tau}, \varphi\right)d\tau$$

The asymptotic form of W_1 for large r, φ is constructed by the stationary-phase method (Section 2). When calculating W_2 we can use the asymptotic form (3.1), (3.2), i.e. we can consider the integral

$$W_2 = \int_0^t h\left(\frac{r}{\tau}\right)C\left(\frac{r}{\tau}\right)\exp i\Phi(r, \varphi, \tau)\frac{d\tau}{\tau}, \quad C\left(\frac{r}{\tau}\right) = C_0 + \frac{r}{\tau}C_1 + \ldots \tag{3.4}$$

4. THE ASYMPTOTIC FORM OF W_2 WHEN THE FUNCTION $\Phi_2(\varphi)$ HAS A LOWER BOUND

If the function $\Phi_2(\varphi)$ has a lower bound in modulus, then for sufficiently large $\tau : \tau > \tau_1$ the derivative $\partial\Phi/\partial\tau$ has a lower bound in modulus of value const r^2/τ^2. In the expansion of unity (3.3) we can assume that $C_1 > \tau_1$. Then the stationary point τ_0 of the phase function is in the region supp $g(r/\tau)$ and the asymptotic form in the far zone of the integral W_1 is identical with the contribution of this point, i.e. is equal to U_1 (see formula (2.2)).

To calculate the asymptotic form of W_2 we change to the variable of integration $\xi = r/\tau$

$$W_2 = \int_{r/t}^\infty \exp(ir\Psi(\xi))h(\xi)C(\xi)\frac{d\xi}{\xi} \tag{4.1}$$

Here $h(\xi)$ is a finite infinitely differentiable function, identically equal to unity in the neighbourhood of zero, the function $C(\xi)$ is the same as in (3.4), while the function

$$\Psi(\xi) = \xi\Phi_2(\varphi) + \xi^2\Phi_3(\varphi) + \ldots$$

in the region supp h has a derivative with a lower bound in modulus. Hence, W_2 when $r \gg 1$ is an integral of a rapidly oscillating function with two close critical points—a pole $\xi = 0$ of the factor outside the exponential function and the boundary $\xi = r/t$ of the integration region. The asymptotic form of these integrals, uniform with respect to the distance between the critical points, is expressed (see, for example, [6]) in terms of the integral of the exponential function E_1 of imaginary argument. We have

$$W_2 \approx C_0E_1(-i\Phi(r,\ \varphi,\ t))+iH(t)\frac{\exp i\Phi(r,\ \varphi,\ t)}{r}+O(r^{-2})$$

$$H(t)=\frac{tC(r/t)}{r\Psi'(r/t)}-\frac{C_0}{\Psi(r/t)}=\frac{C_1\Phi_2(\varphi)-C_0\Phi_3(\varphi)}{\Phi_2^2(\varphi)}+O\left(\frac{r}{t}\right),\quad E_1(\pm iz)=\int_{\pm iz}^{\infty}e^{-\sigma}\frac{d\sigma}{\sigma}$$

where we have assumed that $z > 0$. Since $E_1(\pm iz) \approx \mp i\exp(\mp iz)/z$ when $z \gg 1$, the asymptotic form of W_2 when $|\Phi(r, \varphi, t) \gg 1$ is identical with U_2. Hence, the asymptotic form (2.4) is applicable when $|\Phi(r, \varphi, t)| \gg 1$, i.e. when

$$t \ll r^2 |\Phi_2(\varphi)| = r^2\left|\frac{\cos^2\varphi}{4A}+\frac{\sin^2\varphi}{4B}\right|$$

As $z \to 0$ the function $E_1(\pm iz)$ behaves as $\ln z$. Hence, when t increases the function W_2 increases logarithmically

$$W_2(t,\ r,\ \varphi)=C_0\ln(r^2\Phi_2(\varphi)/t)+H(t)/r+O(r^{-2})+O(t^{-1}) \tag{4.2}$$

5. THE ASYMPTOTIC FORM OF W_2 FOR SMALL $\Phi_2(\varphi)$

If $\Phi_2 \to 0$, i.e. φ approaches arctg $\sqrt{(-B/A)}$ or $\pi \pm$ arctg $\sqrt{(-B/A)}$, the stationary point

$$\tau_0=-2r\Phi_3(\varphi)/\Phi_2(\varphi)+O(1) \tag{5.1}$$

of the phase function Φ approaches infinity. Hence, when calculating the asymptotic form $\widetilde{W}$ in the case of small Φ_2 we can assume that, in the expansion of unity (3.3), the constant C_2 satisfies the condition $C_2 < |\Phi_3(\varphi)/\Phi_2(\varphi)|$. The phase function in the integral of W_1 will have no stationary points in the region supp g, and this integral will decrease as $t \to \infty$ more rapidly than any power of r and the asymptotic form of the field $\widetilde{W}$ will be identical with the asymptotic form of W_2. As can be seen from (4.1), when $t \gg r \gg 1$ this function is the integral of a rapidly oscillating function with three close critical points: the pole $\xi = 0$ of the factor outside the exponential, the boundary $\xi = r/t$ of the integration region and the stationary point $\xi = r/t_0$ of the phase function $\psi(\xi)$, where the second derivative ψ'' of the phase function has a lower bound in modulus. The simplest integral with such critical points is the Ff-integral, introduced in [6]†

$$Ff(\sqrt{r}\alpha,\ \sqrt{r}\beta)=\frac{1}{2\pi}\int_{-\infty}^{\sqrt{r}\alpha}\frac{\exp(is^2)ds}{s-\sqrt{r}\beta+i0}$$

where, when $\alpha > \beta$, the pole $s = \sqrt{(r)}\beta$ is circumvented in the upper half-plane.

The uniform asymptotic form of integral (4.1) when $\Psi'' > 0$, by [6], can be expressed in terms of the Ff-integral and the Fresnel integral using the formula

$$\begin{aligned}W_2 &= 2\pi\exp(ir\Psi(r/\tau_0))C_0Ff(\sqrt{r}\alpha,\sqrt{r}\beta)+\\ &+\sqrt{\frac{\pi}{r}}\,S\exp(ir\Psi(r/\tau_0))\exp(\pi i/4)F(\sqrt{r}\alpha)+\frac{i}{r}T\exp(ir\Psi(r/t))+O(r^{-3/2})\end{aligned} \tag{5.2}$$

$$F(\sqrt{r}\alpha)=\frac{\exp(-\pi i/4)}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{r}\alpha}\exp(is^2)ds$$

where

†See also ANYUTIN, A. P. and BOROVIKOV, V. A., Uniform asymptotic forms of integrals of rapidly oscillating functions with singularities of the factor outside the exponential. Preprint No. 42(414), Inst. of Radioelectronics, Akad. Nauk SSSR, Moscow, 1984.

$$\alpha = \operatorname{sign}(t/\tau_0 - 1)\sqrt{|\Psi(r/t) - \Psi(r/\tau_0)|}, \quad \beta = \operatorname{sign}\tau_0\sqrt{|\Psi(r/\tau_0)|} \tag{5.3}$$

When $\Psi'' < 0$ the function $Ff(\sqrt{(r)}\alpha, \sqrt{(r)}\beta)$ and $\exp(\pi i/4)F(\sqrt{(r)}\alpha)$ in (5.2) can be replaced by the complex conjugates. The quantities α, β, S, T can be expressed in terms of the function Ψ and its derivatives. In particular, retaining the principal terms of the expansions in powers of the small parameter $\Phi_2(\varphi)$, we obtain

$$\alpha = -\frac{r|\Phi_3|^{1/2}}{t} - \frac{\Phi_2|\Phi_3|^{1/2}}{2\Phi_3}, \quad \beta = -\frac{\Phi_2|\Phi_3|^{1/2}}{2\Phi_3}, \quad S = C|\Phi_3|^{-1/2}, \quad T = 0 \tag{5.4}$$

6. THE BEHAVIOUR OF THE FIELD $\hat{W}(t, \hat{x}, \hat{y})$ WHEN $t, r \gg 1$

The behaviour of the field $\hat{W}(t, \hat{x}, \hat{y})$ depends on the type of stationary point $O = (0, 0)$ of the function $\hat{\omega}(\lambda_0 + \eta, \mu_0 + \zeta)$. We will assume first, that this point is a point of extremum, i.e. that in expansion (1.2) the coefficients A and B have the same sign and the point O is an isolated point of the curve $\hat{\omega} = 0$. When $t, r \gg 1$ the far field $W(t, r\cos\varphi, r\sin\varphi)$ then consists of the following components

1. That due to the singular points of the function $\omega(\lambda_0, \mu_0)$ or the factor $F(\lambda, \mu)$ outside the exponential of the component $\Omega(t, r, \varphi)$.

2. The components $U_i(t, r, \varphi)$ due to the regular points P_i of the curve $\hat{\omega} = 0$ in which grad $\hat{\omega}$ has the direction φ. Each term represents a wave propagating in the direction φ with velocity $\upsilon_i(\varphi) = |\text{grad}|\,\hat{\omega}(P_i)$. In front of the wave front, i.e. when $r > \upsilon_i(\varphi)t$, the component U_i is of the order of t^{-1}. In the neighbourhood of the wave front, i.e. when $r \approx \upsilon_i(\varphi)t$, U_i can be expressed in terms of the Fresnel integral (if P_i is a point of inflection of the curve $\hat{\omega} = 0$), while behind the wave front, when $r < \upsilon_i(\varphi)t$ U_i it is independent of the main term of the asymptotic form of t and is of the order of $r^{1/2}$. If P_i is a point of inflection of the curve $\hat{\omega} = 0$, then U_i, when $r \leqslant \upsilon_i t$, has a more complex asymptotic form.

3. The component $V(t, r, \varphi)$, due to the singular point $O = (\lambda_0, \mu_0)$ of the curve $\hat{\omega} = 0$. As can be seen from (4.2), this component is of the order of unity when

$$r \leqslant r_Z = \text{const}\sqrt{t/|\Phi_2(\varphi)|} \tag{6.1}$$

In other words, the presence of an isolated singular point O on the curve $\hat{\omega} = 0$ leads to the occurrence of a resonance zone Z in the neighbourhood of zero, in which the field is of the order of unity. As can be seen from (6.1), the size of this zone increase as $\sqrt{t}$ as $t \to \infty$.

Suppose now that 0 is a point of self-intersection of the curve $\hat{\omega} = 0$, i.e. suppose the coefficients A and B in (1.2) have different signs. We will call the directions φ in which Φ_2 vanishes

$$\varphi_{1,2} = \pm\operatorname{arctg}\sqrt{-B/A}; \quad \varphi_{3,4} = \pi \pm \operatorname{arctg}\sqrt{-B/A} \tag{6.2}$$

the critical directions and we will denote by $\Sigma_1, \ldots, \Sigma_4$ the intervals $|\varphi - \varphi_k| < \delta$, where the constant δ is chosen to be fairly small.

Outside these intervals the function Φ_2 has a lower bound in modulus and the field W consists of components of the three types described above. We will show below that this expansion is applicable over a wider range and that the following assertions hold.

A. The expansion W in the components Ω, U_i, V is applicable outside the neighbourhoods of the critical directions, the boundary of which is defined by the equation

$$|\Phi_2(\varphi)| = Cr^{-1/2}\sqrt{|\Phi_3(\varphi)|} \tag{6.3}$$

where

$$|\varphi - \varphi_k| = Cr^{-1/2}\sqrt{|\Phi_3(\varphi)|}/|\Phi_2'(\varphi_k)| = 2Cr^{-1/2}\sqrt{|AB\Phi_3|} \tag{6.4}$$

The constant C will be determined below. It is natural to call these neighbourhoods transition regions.

B. Outside the transition regions the size of the resonance zone in which the field W is of the order of unity is given, as previously, by relation (6.1). Inside the transition region the size of the resonance zone can be estimated from the expression

$$r \leqslant r_Z \approx \text{const}\, t^{2/3} \,|\, \Phi_3 \,|^{-1/3} \tag{6.5}$$

Before proving assertions A and B, we will give some qualitative corollaries of them.

It follows from estimate (6.1) that the size of the resonance zone Z increases as the direction φ approaches the critical direction, from a value of the order of $t^{1/2}$ outside the sectors Σ_k to a value of the order of $2^{2/3}$ at the boundary of the transition regions, where $|\,\Phi_2(\varphi)\,| = O(r^{-1/2})$. Inside the transition regions, the resonance zone, as will be seen below, is of the order of $t^{2/3}$. These estimates indicate that, as t increases, the resonance zone becomes more and more elongated in the critical directions; its size along these directions is $t^{1/6}$ greater than the size in directions differing from the critical directions.

We will now consider how the waves U_i behave close to the critical directions, i.e. inside the sectors Σ_k. These waves only propagate on one side of each of the critical directions φ_k—in the sector of angles φ for which Φ_2 and Φ_3 have different signs and the stationary point τ_0 defined by (5.1) is positive. The distance $r_i(\varphi)$ of the wave front of the wave U_i from the origin of coordinates is found from the condition $t = \tau_0$, i.e.

$$r = r_i(t, \varphi) = -2t\,\Phi_2(\varphi) / \Phi_3(\varphi)$$

As one approaches the critical direction $|\,\Phi_2(\varphi)\,|$ and $r_i(t, \varphi)$ decrease, and on the boundary of the transition zone, where the function $\Phi_2(\varphi)$ is related to r by Eq. (6.3), we obtain

$$r_i = r_i(t) = (2tC)^{2/3} \,|\, \Phi_3 \,|^{-1/3}$$

where C is the same constant as in (6.3).

Hence, although far from the critical directions the distance r_i from the wave front to the origin of coordinates is of the order of t, i.e. much greater than the size of the resonance zone $r_Z = O(\sqrt{t})$, as one approaches the transition region r_i decreases and r_Z increases, whereas on the boundary of this region r_i and r_z are of the same order of magnitude with respect to t, equal to $t^{2/3}$.

When these quantities become close to one another, i.e. inside the transition regions, one cannot separate the wave U_i from the resonance component V in the field W.

Proof of assertions A and B. Inside the intervals Σ_k the modulus of the function $\Phi_2(\varphi)$ is small, and the asymptotic form (5.2) holds for W_2, where we can use (5.4) for the arguments $\sqrt{(r)}\alpha$, $\sqrt{(r)}\beta$ of the Ff-integral and the Fresnel integral, and also for the amplitudes S and T.

The asymptotic form of W_2 when $r, t \gg 1$ is determined by the corresponding asymptotic expansions of the function $Ff(\sqrt{(r)}\alpha, \sqrt{(r)}\beta)$ (see [6]). If the argument $\sqrt{(r)}\beta$ is fairly large, i.e. the pole $s = \sqrt{(r)}\beta$ is sufficiently far from the stationary point $f = 0$ of the phase function (for practical applications it is sufficient to put $|\,\sqrt{(r)}\beta\,| > C$, where $C \approx 2\pi$), then in the asymptotic form of the Ff-integral we can consider separately the contribution of the stationary point $\sigma = 0$ (when $\alpha > 0$) and the pole $\sigma = \beta$ (as $\alpha \to \beta$). The first term corresponds to the wave U_i, and its wave front has the equation $\alpha = 0$, which, as a consequence of (5.3), is identical with (6.5). The second term, which increases logarithmically as $\alpha \to \beta$, corresponds to the component V.

When $|\,\sqrt{(r)}\beta\,| < C$ we cannot separate the contribution of the stationary point $\sigma = 0$ from the contribution of the pole $\sigma = \beta$ in the asymptotic form of the Ff-integral. The condition $|\,\sqrt{(r)}\beta\,| = C$ defines the boundary of the transition region.

If we use the asymptotic form of the Ff-integral outside the transition region, which can be done when $|\,\sqrt{(r)}\beta\,|$ is sufficiently large (see [6]), formula (5.2) reduces to (4.2), whence assertions A and B follow for points r, φ, which lie outside the transition regions.

It remains to prove that estimate (6.1) holds for the size of the resonance zone inside the transition regions. In these regions, i.e. for bounded $|(r)\beta|$, it is more convenient to use the following expression for the Ff-integral (see [6])

$$Ff(\sqrt{r}\alpha,\ \sqrt{r}\beta) = \exp(i\beta^2)\left[-\frac{1}{4\pi}E_1(ir(\alpha-\beta)^2) - \frac{i}{2}F^*(\sqrt{r}(\alpha-\beta))\right] + J_2(\sqrt{r}\alpha, \sqrt{r}\beta)$$

where F^* is a function which is the complex conjugate of the Fresnel integral, and

$$J_2(\sqrt{r}\alpha, \sqrt{r}\beta) = -\frac{\exp(ir\alpha^2)}{4\pi}\int_0^{\infty} \exp(i\sigma^2)\frac{\exp(-2i\sigma\sqrt{r}\alpha) - \exp(-2ir(\alpha-\beta)\alpha)}{\sigma - \sqrt{r}(\alpha-\beta)}\,d\sigma$$

can be expanded in a convergent series in powers of $\sqrt{(r)}(\alpha - \beta)$, $\sqrt{(r)}\alpha$. Hence it follows that, inside the transition regions, the logarithmically increasing component of the fields W_2 is given by the function

$$\frac{1}{4\pi}E_1(ir(\alpha-\beta)^2)=\frac{1}{4\pi}E_1(ir^3\,|\Phi_3|\,t^{-2})$$

which has an order of magnitude not less than unity for small $r^3|\Phi_3|t^{-2}$. Hence estimate (6.4) also follows.

7. DEGENERACY OF THE STATIONARY POINT $\eta = \zeta = 0$ OF THE FUNCTION $\tilde{\omega}$

In the previous analysis we assumed that the stationary point $\eta = \zeta = 0$ of the function $\hat{\omega}(\lambda_0 + \eta, \mu_0 + \zeta)$ is non-degenerate, i.e. that the coefficients A and B do not vanish in expansion (1.2). This condition is not always satisfied. We will estimate the size of the resonance zone and the order of increase in W_2 as $t \to \infty$ for the case when the stationary point is degenerate. To fix our ideas we will put $A = 0$ in expansion (1.2).

In subsequent calculations we will omit terms which have no effect on the final estimates.

Assuming $G = 1$ in (1.3) we carry out the integration over ζ in the inner integral and we express the integral over η in terms of the Airy function. It is convenient to separate the dimensional factor $|B|^{-1}$ from the function $\tilde{W}$

$$\tilde{W}=\overline{W}/|B|$$

Then $\overline{W}$ will be a dimensionless function of $t, \hat{x}, \hat{y}$

$$\overline{W}=\frac{2\pi^{3/2}\sqrt{|B|}}{|3C|^{1/3}}\exp(i\pi\,\mathrm{sign}\,B/4)\int_0^t \exp(iS(\tau))Ai\left(\frac{r\cos\varphi}{(3C\tau)^{1/3}}+\frac{qr^2\sin^2\varphi}{\tau^{4/3}}\right)\frac{d\tau}{\tau^{5/6}}$$

$$S(\tau)=-\frac{r^2\sin^2\varphi}{4B\tau}+\frac{D^3r^3\sin^3\varphi}{72B^3C^2\tau^2}+\frac{Dr^2\cos\varphi\sin\varphi}{6BC\tau}$$

$$p=(3C)^{-1/3},\quad q=\frac{D^2}{4B^2(3C)^{4/3}}$$

where r, φ are the polar coordinates (2.1). Since the critical directions φ_k, by (6.2), approach $\pm\pi/2$ as $A \to 0$, when $A = 0$ the critical directions are the directions $\pm\pi/2$.

We will estimate the asymptotic form of $\overline{W}$ and the size of the resonance zone for fixed $\varphi \neq \pm\pi/2$ and when $\varphi = \pm\pi/2$. More exactly, we will estimate, for fixed M, the dimensions of the neighbourhood of zero inside which $|\overline{W}| \geqslant M$.

When $\varphi \neq \pi/2$ and $r \geqslant$, $t \geqslant qr\sin^2\varphi/(p\cos\varphi)$ we can neglect the second term in the argument of the Airy function. Changing to the integration variable $\xi = |p\hat{x}\tau^{-1/3}|$, we obtain

$$\overline{W}=\frac{6\pi^{3/2}\sqrt{|B|}\exp[\pi i\,\mathrm{sign}\,B/4]t^{1/6}}{|3C|^{1/3}}H(-(3C)^{-1/3}t^{-1/3}\hat{x})$$

$$H(\theta)=\sqrt{|\theta|}\int_{|\theta|}^{\infty}\exp\{iS[(p\hat{x}/\xi)^3]\}Ai(-\xi\,\mathrm{sign}\,\theta)\frac{d\xi}{\xi^{3/2}}$$

Since $H(\theta)$ approaches a finite limit as $\theta \to 0$, the function $\overline{W}$ increases as $t^{1/6}$ as $t \to \infty$.

We will estimate the size of the resonance zone, i.e. the region in which $\overline{W} \geqslant M$. This region is determined by the values of $\hat{x}$ for which

$$H(-(3C)^{-1/3}t^{-1/3}\hat{x})\sim M|C|^{1/3}|B|^{-1/2}t^{-1/6}\geqslant 1$$

Using the asymptotic form of the Airy function when $|\xi| \gg 1$, we obtain that when $|\theta| \gg 1$

$$H(\theta)\approx\begin{cases}\dfrac{|\theta|^{-7/4}}{\sqrt{\pi}}, & \theta<0\\[2ex] \dfrac{\theta^{-7/4}}{2\sqrt{\pi}}\exp\left(-\dfrac{2}{3}\theta^{3/2}\right), & \theta>0\end{cases}$$

Hence, for fixed $\varphi \neq \pm\pi/2$ and $r \gg 1$ we obtain for the size of the resonance zone

$$|\hat{x}_z| = r_z |\cos\varphi| \approx \begin{cases} 6M^{-4/7}B^{2/7}C^{1/7}t^{3/7} & \text{when } \operatorname{sign}\hat{x} = \operatorname{sign} C \\ (3C)^{1/3}t^{1/3} & \text{when } \operatorname{sign}\hat{x} = -\operatorname{sign} C\,) \end{cases} \tag{7.1}$$

This asymptotic form is inapplicable when $\hat{y} \gg \hat{x}$. We will estimate the size of the resonance zone when $\hat{x} = 0$. This estimate has a different form when $D \neq 0$ and $D = 0$. Suppose first that $D \neq 0$. Then, changing to the integration variable $\xi = q\hat{y}^2\tau^{-4/3}$ we obtain

$$\overline{W} = \frac{3\pi^{3/2}\sqrt{|B|}}{2|3C|^{1/3}} \exp\left(\frac{\pi i \operatorname{sign} B}{4} t^{1/6} T(q\hat{y}^2 t^{-4/3})\right)$$

$$T(\theta) = |\theta|^{1/8} \int_{|\theta|}^{\infty} e^{iS[(q\hat{y}^2/\xi)^{3/4}]} \operatorname{Ai}(\xi \operatorname{sign}\theta) \frac{d\xi}{\xi^{9/8}}$$

It can be seen that on the $\hat{y}$ axis the order of increase of $\overline{W}$ as $t \to \infty$ is the same as when $\hat{x} \gg 1$ and has a lower bound $|\hat{x}/\hat{y}|$.

We will estimate the size of the resonance zone along the $\hat{y}$ axis. When $|\theta| \gg 1$ the function $T(\theta)$ and has the same asymptotic form as the function $H(\theta)$. Hence, we obtain for the size of the resonance zone along the $\hat{y}$ axis, i.e. when $\varphi = \pm\pi/2$

$$r_Z = |\hat{y}| \sim \begin{cases} 1{,}2M^{-2/7}|B|^{1/7}|C|^{-2/21}|q|^{-1/2}t^{5/7}, & q < 0 \\ |q|^{-1/2}t^{2/3}, & q > 0 \end{cases} \tag{7.2}$$

When $D = 0$ and $x = 0$ we have

$$\overline{W} = \frac{2\pi^{3/2}\sqrt{|B|}\exp[\pi i \operatorname{sign} B/4]\operatorname{Ai}(0)t^{1/6}}{|3C|^{1/3}} T\left(\frac{y^2}{4Bt}\right)$$

$$T(\theta) = |\theta|^{1/6} \int_{|\theta|}^{\infty} \exp[-i\xi \operatorname{sign}\theta] \frac{d\xi}{\xi^{7/6}}$$

The function $|T(\theta)| \approx |\theta|^{-1}$ when $|\theta| \gg 1$, and hence we obtain the following relation for the size of the resonance zone along the $\hat{y}$ axis

$$r_Z = |\hat{y}| \sim 2\sqrt{2\operatorname{Ai}(0)}(3C)^{-1/6}(\pi|B|)^{3/4}M^{-1/2}t^{7/12} \tag{7.3}$$

As was shown in Section 6, for the case of a non-degenerate stationary point the size of the resonance zone in the critical direction was $t^{1/6}$ times greater than in a direction differing from the resonance direction. It can be seen from the estimates obtained that for the case of a degenerate stationary point, when two critical directions merge, the ratio of these dimensions increases more rapidly—as $t^{2/7}$.

8. THE THREE-DIMENSIONAL PROBLEM

We will outline the construction of the asymptotic form of the far field in the case of resonance for the three-dimensional problem. In the three-dimensional case the analogue of (1.3) is the expression

$$\tilde{W}(t,\hat{x},\hat{y},\hat{z}) = \int_0^t J d\tau$$

$$J = \iiint G(\eta,\zeta,\gamma)\exp i(-\eta\hat{x} - \zeta\hat{y} - \gamma\hat{z} + \tau\tilde{\omega}(\eta,\zeta,\gamma))d\eta d\zeta d\gamma \tag{8.1}$$

where

$$\tilde{\omega}(\eta,\zeta,\gamma) = \omega(\lambda_0+\eta, \mu_0+\zeta, \xi_0+\gamma) - \omega(\lambda_0,\mu_0,\xi_0) - \eta V_x - \zeta V_y - \gamma V_z$$

The integral $J = J(\tau, r, \theta, \varphi)$ in (8.1) (r, θ, φ are spherical coordinates in the space $\hat{y}, \hat{z}$) is of the order of $\tau^{-3/2}$ as $\tau \to \infty$ (see below). Hence, unlike the two-dimensional case the external integral converges as $t \to \infty$, i.e. a steady state

$$W(r,\theta,\varphi) = \int_0^\infty J(\tau,r,\theta,\varphi)d\tau \tag{8.2}$$

exists.

In the case of resonance the function $\tilde{\omega}(\eta, \zeta, \gamma)$ vanishes at the origin of coordinates together with its first derivatives. We will assume that the matrix of the second derivatives at this point is non-degenerate. Then, the directions of the x, y, z axes can be chosen in such a way that the expansion of $\tilde{\omega}(\eta, \zeta, \gamma)$ in powers of $\tilde{\omega}(\eta, \zeta, \gamma)$ has the form

$$\tilde{\omega}(\eta,\zeta,\gamma) = \frac{a\eta^2}{2} + \frac{b\zeta^2}{2} + \frac{c\gamma^2}{2} + \omega_3(\eta,\zeta,\gamma) + \dots$$

where ω_3 is a homogeneous polynomial of the third degree, and the dots denote the following terms of the expansion.

We will also assume that the function G vanishes outside a fairly small neighbourhood of the origin of coordinates. Under these conditions, the asymptotic form of $J(\tau, r, \theta, \varphi)$ when $\tau \gg r \gg 1$ can be calculated by the usual stationary-phase method

$$J(\tau,r,\theta,\varphi) \approx \frac{\text{const}}{r^{3/2}} \exp\left[-ir\left(\frac{r}{\tau}\Phi_2 + \frac{r^2}{\tau^2}\Phi_3 + \frac{r^3}{\tau^3}\Phi_4 + O\left(\frac{r^4}{\tau^4}\right)\right)\right] \tag{8.3}$$

Here

$$\Phi_k = -(-1)^k \omega_k\left(\frac{\cos\theta}{2a}, \frac{\sin\theta\cos\varphi}{2b}, \frac{\sin\theta\sin\varphi}{2c}\right), \quad k = 2,3$$

and Φ_4 is a homogeneous polynomial of the fourth degree of $\cos\theta$, $\sin\theta$, $\cos\varphi$, $\sin\theta\sin\varphi$.

Using the expansion of unity (3.3), we can reduce the problem to calculating the asymptotic form of the integral of the product $h(r/\tau)J(\tau, r, \theta, \varphi)$, where h is a cutoff function, which separates the neighbourhood of an infinitely distant point at which the asymptotic form (8.3) is applicable for J. Making the change of variable $\xi = \sqrt{r/\tau}$ we obtain

$$W_2(r,\theta,\varphi) \approx \frac{\text{const}}{r^{1/2}} \int_{-\infty}^{\infty} h(\xi^2)\exp ir(\xi^2\Phi_2 + \xi^4\Phi_3 + \xi^6\Phi_4 + \dots)d\xi$$

The asymptotic form of W_2 as $r \to \infty$ is determined by the values of the functions Φ_2, Φ_3, Φ_4. If the origin of coordinates $O = (0, 0, 0)$ is a point of extremum of the function $\tilde{\omega}$, i.e. the coefficients a, b and c in (8.3) have the same sign, then $\Phi_2(\theta, \varphi)$ has a lower bound in modulus for any θ, φ. Hence, W_2 is of the order of r^{-1} as $r \to \infty$, i.e. of the same order as the components of the far field, due to regular points of the surface $\tilde{\omega} = 0$ at which grad $\tilde{\omega}$ has the direction θ, φ (see [1, 2]). If O is a saddle point of the function $\tilde{\omega}$, i.e. a conical point of the surface $\tilde{\omega} = 0$, then the asymptotic form of W_2 for fixed θ, φ and as $r \to \infty$ is of the order of r^{-1} when $\Phi_2 \neq 0$, $r^{-3/4}$ when $\Phi_2 = 0$, but $\Phi_3 \neq 0$ and $r^{-2/3}$ when $\Phi_2 = \Phi_3 = 0$, but $\Phi_4 \neq 0$.

We will explain the geometrical meaning of these conditions.

The condition $\Phi_2 = 0$ defines the cone S of critical directions θ, φ for which the plane Σ $\eta\cos\theta + \zeta\sin\theta\cos\varphi + \gamma\sin\theta\sin\varphi = 0$ is touched at the point O by the surface $\tilde{\omega} = 0$. If Φ_3 changes sign on the cone S, then there are directions θ_i, φ_i on this cone for which $\Phi_3 = 0$. We will call these directions supercritical directions. For these directions the plane Σ comes in contact with the surface $\tilde{\omega} = 0$ at the point O. In general, the function Φ_4 does not vanish in supercritical directions.

Hence, W_2 is of the order of r^{-1} for directions θ, φ which differ from the critical directions, is of the order of $r^{-3/4}$ for critical directions which differ from the supercritical directions, and is of the order of $r^{-2/3}$ for supercritical directions.

This asymptotic form is non-uniform, i.e. it is inapplicable for directions θ, φ close to the critical and supercritical directions respectively. In the first case, i.e. for small Φ_2 and for Φ_3 having a lower bound in modulus, the model integral describing the uniform asymptotic form W_2 is the integral

$$J_2 = r^{-3/4} |\Phi_3|^{-1/4} \int_{-\infty}^{\infty} \exp i(\alpha\xi^2 + \xi^4 \operatorname{sign}\Phi_3)d\xi, \quad \alpha = r^{1/2}\Phi_2 |\Phi_3|^{-1/2}$$

which reduces to the Pearcey integral [7].

Similarly, in the neighbourhood of a supercritical direction, i.e. directions in which Φ_2 and Φ_3 are small and the function Φ_4 has a lower bound in modulus, the model integral is the integral

$$J_3 = r^{-2/3} |\Phi_4|^{-1/4} \int_{-\infty}^{\infty} \exp i(\alpha\xi^2 + \beta\xi^4 + \xi^6 \operatorname{sign} \Phi_4) d\xi$$

$$\alpha = r^{2/3} \Phi_2 |\Phi_4|^{-1/3}, \quad \beta = r^{1/3} \Phi_3 |\Phi_4|^{-2/3}$$

It reduces to the generalized Airy functions introduced in [8].

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